QUIZ 3 - CALCULUS 2 (2020/12/10)

Evaluate the following indefinite integrals.

1. (5 pts) $\int \tan^5 x\, dx$.

   Solution:

   Method 1

$$\int \tan^5 x dx = \int \tan^4 x \tan x dx = \int (\sec^2 x - 1)^2 \tan x dx$$
$$= \int (\sec^4 x - 2\sec^2 x + 1) \tan x dx$$
$$= \int \sec^3 x \sec x \tan x dx - 2\int \sec x \sec x \tan x dx + \int \tan x dx$$
$$=(*) \quad \color{red}{(1\,pt)}$$
$$\int \sec^3 x \sec x \tan x dx \overset{y=\sec x}{=} \int y^3 dy = \frac{1}{4}\sec^4 x + C \quad \color{red}{(1\,pt)}$$
$$\int \sec x \sec x \tan x dx \overset{y=\sec x}{=} \int y dy = \frac{1}{2}\sec^2 x + C \quad \color{red}{(1\,pt)}$$
$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx \overset{y=\cos x}{=} -\int \frac{1}{y} dy = -\ln|\cos x| + C \quad \color{red}{(1\,pt)}$$

Hence

$$(*) = \frac{1}{4}\sec^4 x - \sec^2 x - \ln|\cos x| + C. \quad \color{red}{(1\,pt)}$$

Method 2

$$\int \underbrace{\tan^5 x}_{\tan^2 x \tan^3 x} dx = \int (\sec^2 x - 1)\tan^3 x\, dx \quad \color{red}{(1\,pt)}$$
$$= \int \sec^2 x \tan^3 x\, dx - \int \underbrace{\tan^3 x}_{\tan^2 x \tan x} dx$$
$$= \int \sec^2 x \tan^3 x\, dx - \int (\sec^2 x - 1)\tan x\, dx \quad \color{red}{(1\,pt)}$$
$$= \int \sec^2 x \tan^3 x\, dx - \int \sec^2 x \tan x\, dx + \int \tan x\, dx$$
$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C \quad \color{red}{(3\,pt)}$$

2. (10 pts) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx.$

Solution:

Use the trig. substitution. Let $x = \sin\theta$, where $\theta \in (-\pi/2, \pi/2)$. Then $dx = \cos\theta d\theta$ and $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$ since $\cos\theta > 0$ when $\theta \in (-\pi/2, \pi/2)$. (2 pt)

So the integral becomes

$$\int \frac{x^2}{\sqrt{1-x^2}}dx = \int \frac{\sin^2\theta}{\cos\theta}\cos\theta d\theta = \int \sin^2\theta d\theta$$
$$= \int \frac{1-\cos(2\theta)}{2}d\theta \quad (2\,pt)$$
$$= \frac{\theta}{2} - \frac{\sin(2\theta)}{4} + C \quad (2\,pt)$$
$$= \frac{\sin^{-1}x}{2} - \frac{2\sin\theta\cos\theta}{4} + C$$
$$= \frac{\sin^{-1}x}{2} - \frac{x\sqrt{1-x^2}}{2} + C \quad (4\,pt).$$

3. (5 pts) $\displaystyle\int x\sin^{-1}x\,dx$ (Hint: you can use (2)).

Solution:

Use the integration by parts.

$$\int x\sin^{-1}x dx \quad (du = xdx, v = \sin^{-1}x, \text{ then } u = \frac{x^2}{2}, dv = \frac{1}{\sqrt{1-x^2}}dx) \quad (2\,pt)$$
$$= \frac{x^2\sin^{-1}x}{2} - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx \quad (2\;pt)$$
$$= \frac{x^2\sin^{-1}x}{2} - \frac{\sin^{-1}x}{4} + \frac{x\sqrt{1-x^2}}{4} + C \quad (1\,pt).$$

The last integral is obtained from (2). So if a student makes mistakes in (2), we take only one point off here.